Artin Wedderburn

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1 Current division of team work

Summary: we have defined the necessary structures that are used in the proof. Namely, ideal products, the corner ring, and matrix units. We have proved two major auxiliary theorems that are used in the proof of the main theorem: Brauer's lemma and theorem 26. Our next target is theorem 25 which is the last prerequisite of the main theorem. After that, we will split the proof of the main theorem into manageable parts.

- **Matevž Miščič:** Has done some major parts of the proof: theorem 6, properties of ideal products and set-like products, properties of corner rings, Brauer's lemma, theorem 19, theorem 24, and theorem 11. Now he is helping the new members.
- **Maša Žaucer:** Assigned and working on theorem 17 and theorem 25 as an introduction to the project. After that, she will split the proof of the main theorem 28. This will include setting up the definitions of subrings for which the conclusion of the statement holds. A handful of intermediary lemmas will be needed to enable the use of the artinian property.
- **Job Petrovčič:** Has done some major parts of the proof: initial project setup, definition of ideal products, definition of corner ring, basic properties of corner rings, definition of matrix units, and theorem 26. Now he is helping the new members.

If time permits, we will work on the uniqueness part of the proof.

2 Preliminaries

Definition 1 (Set aRb)**.** For $a, b \in R$, denote by aRb the set $\{arb | r \in R\}$.

Definition 2 (Left/Right Ideal)**.** A *left/right ideal* I of a ring R is an additive subgroup of R such that $rI \subseteq I$ for all $r \in R$ or $Ir \subseteq I$ for all $r \in R$, respectively.

Definition 3 (Two-sided Ideal)**.** A two-sided ideal is a subset of R that is both left and right ideal of R.

Definition 4 (Product of Ideals)**.** A product of (left/right/two-sided) ideals I and J is the ideal IJ generated by the set of all pairwise products of elements of I and J.

Definition 5 (Prime Ring)**.** A ring is prime if we have $I = 0$ or $J = 0$ whenever $IJ = 0$ for some left ideals I and J.

Theorem 6. *A ring is prime if and only if for all $a, b \in R, aRb = 0$ implies $a = 0$ or $b = 0$.*

Proof. ($\Rightarrow$) Suppose $aRb = 0$. Then $(Ra)(Rb) = 0$, thus by primality $Ra = 0$ or $Rb = 0$. In the former case we get $a = 1 \cdot a \in Ra$ and thus $a = 0$ and in the latter case we get $b = 1 \cdot b \in Rb$ and thus $b = 0$.

($\Leftarrow$) Suppose $IJ = 0$. Then $aRb \subseteq IJ = 0$ for any $a \in I$ and $b \in J$. By assumption we get $a = 0$ or $b = 0$, so at least one of I and J is zero. □

Theorem 7. *A ring is prime if and only if for all two-sided ideals I and J, $IJ = 0$ implies $I = 0$ or $J = 0$.*

Proof. ($\Rightarrow$) Two-sided ideals are left ideals, so the result follows directly from definiton.

($\Leftarrow$) Suppose $aRb = 0$. Then $(RaR)(RbR) = 0$. By assumption $RaR = 0$ or $RbR = 0$. Thus $a = 0$ or $b = 0$ so the result follows from the previous theorem. □

Definition 8 (Simple Ring)**.** A ring is simple if it has no nontrivial two-sided ideals.

Theorem 9. *A simple ring is prime.*

Proof. Suppose $IJ = 0$. If both I and J are nonzero, they must be equal to R by simplicity. But $RR = R \neq 0$, a contradiction. □

Definition 10 (Orthogonal Elements)**.** Two elements $a, b \in R$ are orthogonal if $ab = ba = 0$.

3 Proof of Artin-Wedderburn Theorem for prime and simple rings

The proof is heavily based on [1].

Theorem 11. *If $e, f \in R$ are orthogonal idempotents and $f \neq 0$, then the left ideal generated by $1 - e - f$ is strictly smaller than the left ideal generated by $1 - e$.*

Proof. Note that $(1-e-f)(1-e) = 1-e-f$, and hence $x(1-e-f) = x(1-e-f)(1-e) \in R(1-e)$ for every $x \in R$. This proves that $R(1-e) \supseteq R(1-e-f)$.

We have $f = f(1-e) \in R(1-e)$, while $f = x(1-e-f)$ with $x \in R$ implies $0 = f(1-f) = x(1-e-f)(1-f) = x(1-e-f) = f$, a contradiction. Therefore, $R(1-e) \neq R(1-e-f)$. □

From here on, let e and f denote orthogonal idempotents in R.

Definition 12 (Corner Ring Set)**.** The set of the corner ring is eRe.

Theorem 13. *An element x is in the set eRf if and only if $x = exf$.*

Proof. ($\Rightarrow$) Suppose $x \in eRf$. Then $x = eyf$ for some $y \in R$. But then $exf = eeyff = eyf = x$.

($\Leftarrow$) Clear. □

Theorem 14. *An element x of R is in the corner ring if and only if $x = exe$.*

Proof. Application of theorem 13. □

Theorem 15. *An element x of the corner ring is of the form eye for some $y \in R$.*

Proof. Clear from the theorem 14 □

Theorem 16. *The corner ring is a (non-unital) subring of R. It has its own unit e.*

Proof. If $a, b \in eRe$, then $a + b = eae + ebe = e(a + b)e$ so eRe is closed under addition. If $a, b \in eRe$, then $ab = eaeebe = eabe$, so eRe is closed under multiplication. Distributivity and associativity are inherited from R.

Since $ea = eeae = eae = a = eae = eaee = ae$ for any $a \in eRe$, e is the unit of eRe. □

Theorem 17. *If R is left artinian, then the corner ring is left artinian.*

Proof. Let $L_1 \supseteq L_2 \supseteq ...$ be a descending chain of left ideals in eRe. Then $RL_1 \supseteq RL_2 \supseteq ...$ is a descending chain of left ideals in R. Since R is left artinian, this chain stabilizes. But then so does $eRL_1 \supseteq eRL_2 \supseteq ....$ But since $eRL_i = eReL_i = L_i$, the chain $L_1 \supseteq L_2 \supseteq ...$ also stabilizes. □

Theorem 18. *If R is a prime ring, then the corner ring is prime.*

Proof. Suppose $aeReb = 0$ for $a, b \in eRe$. Then $ae = a = 0$ or $eb = b = 0$ by 6, and by the same theorem, the ring is prime. □

Theorem 19. *If all elements in a ring are left invertible, then the ring is a division ring.*

Proof. Let $x \in R$ be arbitrary. Then $yx = 1$ for some $y \in R$. Since y is left invertible, there exists some z such that $zy = 1$. By uniqueness of left and right inverses of y it must hold that $z = x$. Thus x is invertible. □

Theorem 20 (Brauer's lemma)**.** *Suppose L is a minimal (left) ideal of R and $L^2 \neq 0$. Then there exists an idempotent $e \in L$ such that $L = Re$ and eRe is a division ring.*

Proof. By assumption, there exists $y \in L$ such that $Ly \neq 0$. By minimality $L = Ly$. Thus, there exists $e \in L$ such that $ey = y$. Let $J \subseteq L$ be the set of elements in L that annihilate y from the left.

Claim 1. *J is a left ideal of R contained in L.*

Proof. Let $a, b \in J$. Then $(a + b)y = ay + by = 0$, so $(a + b) \in J$. For any $x \in R$, $xay = 0$ so $xa \in J$. □

The element e is not in J, therefore $J = 0$ by minimality of L. Rearranging the previous equality, $(e^2 - e)y = 0$ which implies $e^2 = e$, since $e^2 - e$ is in $J = 0$. Clearly $e \neq 0$, and so by minimality $Re = L$.

Let $a \in eRe$ be non-zero. Then $0 \neq Ra = Reae \leq Re = L$, so $Ra = L$. Thus $e \in Ra$, so $e = ra$ for some $r \in R$. Then $e = e^2 = erea$, so a is invertible in eRe. We are done by 19 □

Theorem 21. *(Already proven in Mathlib) A nonzero left artinian ring has a minimal left ideal.*

Proof. If minimal left ideal does not exist, then starting with any nonzero left ideal, we can allways find a strictly smaller nonzero left ideal. We can thus construct an infinite strictly decreasing sequence of left ideals, which contradicts the left artinian property. □

Definition 22 (Set of Matrix Units)**.** A set e_{ij} for $i, j \in [1, n]$ is a *set of matrix units* of R if

$$e_{ij}e_{kl} = \begin{cases} e_{il} & | \; j = k \\ 0 & | \text{ otherwise} \end{cases}$$

and $\sum_{i=1}^{n} e_{ii} = 1$.

Theorem 23. *If R has a set of matrix units e_{ij}, then R is isomorphic to the ring of $n \times n$ matrices over the corner ring $e_{11}Re_{11}$.*

Proof. For $a \in R$, denote $a_{ij} = e_{1i}ae_{j1}$. Then $e_{11}a_{ij}e_{11} = e_{11}e_{1i}ae_{j1}e_{11} = e_{1i}ae_{j1}$ by the property of matrix units. Then, the map ϕ claimed to be the isomorphism is $a \mapsto (a_{ij})_{i,j=1}^n$.

Claim 2. *ϕ is additive.*

Proof. For $a, b \in R$, we have: $((a+b)_{ij})_{i,j=1}^n = (e_{1i}(a+b)e_{j1})_{i,j=1}^n = (e_{1i}ae_{j1} + e_{1i}be_{j1})_{i,j=1}^n = (a_{ij} + b_{ij})_{i,j=1}^n$ □

Claim 3. *The map is multiplicative.*

Proof. The (i,j) entry of $\phi(a)\phi(b)$ is equal to

$$\sum_{k=1}^n e_{1i}ae_{k1}e_{1k}be_{j1} = e_{1i}a\sum_{k=1}^n e_{kk}be_{j1} = e_{1i}abe_{j1},$$

which is the (i,j) entry of $\phi(ab)$. Therefore, $\phi(ab) = \phi(a)\phi(b)$. □

Claim 4. *The map is injective.*

Proof. Suppose $a_{ij} = 0$ for all i, j. Then $e_{ii}ae_{jj} = e_{1i}a_{ij}e_{j1} = 0$. Therefore, $a = a(\sum_{i=1}^n e_{ii}) = \sum_{i=1}^n ae_{ii} = \sum_{i,j=1}^n e_{ii}ae_{jj} = 0$. □

Claim 5. *The map is surjective.*

Proof. Note the $\phi(e_{k1}ae_{1l})_{kl} = e_{1k}e_{k1}ae_{1l}e_{l1} = e_{11}ae_{11}$ and $\phi(e_{k1}ae_{1l})_{ab} = e_{1a}e_{k1}ae_{1l}e_{b1} = 0$ if $a \neq k$ or $b \neq l$, so $\phi(e_{k1}ae_{1l})$ is a matrix whose all entries are zero, except the k-th and l-th entry is non-zero, and can take arbitrary value in $e_{11}ae_{11}$. By additivity, the map is surjective. □

□

Theorem 24. *If a ring R has a set of pairwise orthogonal idempotents e_{ii} and*

- *$e_{1i} \in e_{11}Re_{ii}$ for all i,*
- *$e_{e1} \in e_{ii}Re_{11}$ for all i,*
- *$e_{1i}e_{e1} = e_{11}$*
- *$e_{i1}e_{1i} = e_{ii}$ for all i,*

then R has matrix units.

Proof. Define $f_{ij} = e_{i1}e_{1j}$.

Claim 6. *For $i = 1$, we have $f_{1j} = e_{1j}$.*

Proof. $f_{1j} = e_{11}e_{1j}$. Since $e_{1j} \in e_{11}Re_{jj}$, we have $e_{11}e_{1j} = e_{1j}$ by theorem 13. □

Claim 7. *For $j = 1$, we have $f_{i1} = e_{i1}$ for all i.*

Proof. $f_{i1} = e_{i1}e_{11}$. Since $e_{i1} \in e_{ii}Re_{11}$, we have $e_{i1}e_{11} = e_{i1}$. □

Claim 8. *$f_{1j}f_{k1} = \delta_{jk}f_{11}$ for all j, k*

Proof. $f_{1j}f_{k1} = e_{11}e_{1j}e_{k1}e_{11} = e_{1j}e_{k1} = e_{11}re_{jj}e_{kk}r'e_{11} = \delta_{jk}e_{11}$ for some r, r', where the last equality comes from the assumption that the diagonal elements are pairwise orthogonal. □

Claim 9. $f_{ij}f_{kl} = \delta_{jk}f_{il}$.

Proof. By definition, $f_{ij}f_{kl} = e_{i1}e_{1j}e_{k1}e_{1l} = f_{i1}f_{1j}f_{k1}f_{1l} = \delta_{jk}f_{i1}f_{1l} = \delta_{jk}e_{i1}e_{1l} = \delta_{jk}f_{il}$ by the previous claims. □

□

Theorem 25. *Let $e, f \in R$ be nonzero orthogonal idempotents and R a prime ring. Also let eRe and fRf be division rings.*

Then there exist $u, v \in R$ such that $u \in eRf$ and $v \in fRe$ such that $uv = e$ and $vu = f$.

Proof.

Claim 10. *There exists $a, b \in R$ such that $eafbe \neq 0$.*

Proof. Suppose $eRf = 0$. By theorem 7, $eRf = 0$ implies $e = 0$ or $f = 0$, a contradiction. Therefore, there exists a such that $eaf \neq 0$.

Suppose $eafRe = 0$. Then $e = 0$ by theorem 7, a contradiction. Therefore, there exists b such that $eafbe \neq 0$. □

Since eRe is a division ring, there exists $c \in R$ such that $(eafbe)(ece) = e$. Let $u = eaf$ and $v = fbece$, which belong to eRf and fRe respectively. Then $uv = eafbece = e$.

Note that $vu \in fRf$ and that $vuv = ve = v = fv$. Therefore, $(vu - f)v = 0$

Claim 11. $vu = f$.

Proof. Suppose not. Then $vu - f \neq 0$, but $vu - f$ is left invertible since fRf is a division ring. Multiplying by the left inverse, we get $v = 0 = fv$, a contradiction with the fact that $uv = e \neq 0$. □

□

Theorem 26. *If a prime ring R contains pairwise orthogonal idempotents e_{ii} with sum 1 and $e_{ii}Re_{ii}$ is a division ring for every i, then R is isomorphic to $M_n(e_{11}Re_{11})$.*

Proof. Applying the theorem 25 for e_{11} and each e_{ii}, we define $e_{1i} = u_i$ and $e_{i1} = v_i$ for each i wher u_i and v_i correspond to u and v in the theorem.

Claim 12. *The defined elements satisfy the conditions of theorem 24.*

Proof. By the conclusion of theorem 25, $e_{1i}e_{i1} = e_{ii}$ and $e_{i1}e_{1i} = e_{11}$ for all i, and $e_{1i} \in e_{11}Re_{ii}$ and $e_{i1} \in e_{ii}Re_{11}$. The e_{ii} are pairwise orthogonal by assumption. □

By theorem 24, R has matrix units, and by theorem 23 it is isomorphic to $M_n(e_{11}Re_{11})$. □

Theorem 27. *If $e, f \in R$ are idempotents and $f \in (1-e)R(1-e)$ they are orthogonal. Further $fRf = f(1-e)R(1-e)f$.*

Proof. $f = f(1-e) + fe = f + fe$. Thus $fe = 0$. Similarly, $ef = 0$. Therefore, f and e are orthogonal.

Note that $x \in fRf \iff \exists r, x = frf = f \iff \exists r, x = f(1-e)r(1-e)f \iff x \in f(1-e)R(1-e)f$. □

Theorem 28 (Artin Wedderburn for prime rings)**.** *If R is a prime ring and artinian, then R is isomorphic to $M_n(D)$ for some division ring D.*

Proof. Since R is artinian, it contains a minimal nonzero left ideal L. If $L^2 = 0$, this would imply by the prime condition that $L = 0$, a contradiction. Therefore, $L^2 \neq 0$. By the Brauer lemma, there exists an idempotent $e \in L$ such that $L = Re_{11}$ and $e_{11}Re_{11}$ is a division ring. By theorem 11 for $e = 0$ and $f = e_{11}$ we have that $R \supsetneq R(1 - e_{11})$.

Suppose $e_{11} \neq 1$. Then $(1 - e_{11})R(1 - e_{11})$ is a nonzero ring. It is also prime and artinian by theorems 18 and 17. Repeating the argument for this ring, we obtain e_{22} such that $e_{22}(1 - e_{11})R(1 - e_{11})e_{22}$ is a division ring. Since $e_{22} \in (1 - e_{11})R(1 - e_{11})$ then must be orthogonal, as by the theorem 27. Further $R(1 - e_{11}) \supsetneq R(1 - e_{11} - e_{22})$. Repeating this process, we get a sequence of e_{ii} and a sequence of left ideals $R(1 - e_{11} - ... - e_{ii})$. By the artinian condition, this sequence must stabilize, so for some n, meaning that $\sum_{i=1}^{n} e_{ii} = 1$. e_{ii} are pairwise orthogonal and are idempotent. Additionally, all $e_{ii}Re_{ii}$ are division rings. By theorem 26, R is isomorphic to $M_n(e_{11}Re_{11})$. □

Theorem 29 (Artin Wedderburn for simple rings)**.** *If R is a simple ring, then R is isomorphic to $M_n(D)$ for some division ring D.*

Proof. Since R is simple, it is prime. By theorem 28, R is isomorphic to $M_n(D)$ for some division ring D. □

4 Generalization to semisimple ring

In this section, we prove the following result, which clearly generalizes Artin Wedderburn to semisimple rings

Theorem 30. *Let R be a semisimple ring. Then, R is isomorphic to a direct product of simple, artinian rings.*

Proof. WLOG, suppose, R is not simple. We know that R is (left) artinian, which is a stronger condition that being (two-sided) artinian. Since it is (two-sided) artinian, it must contain a nontrivial minimal (two-sided) ideal I, which is therefore simple. Since R is semisimple, I must be a direct summand of R (AS A LEFT R-module). Thus, $R = I \oplus J$ for some (left) ideal J. Then $1 = i + j$ for some $i \in I$ and $j \in J$. Note that I and J are both nontrivial.

Claim 13. $IJ = 0$.

Proof. Suppose $x \in IJ$. Then $x \in I$ since I is a twosided ideal. Also $x \in J$ since J is a left ideal. But then $x = 0$ since $I \cap J = 0$. □

Claim 14. *i is an idempotent.*

Proof. $i = i1 = i(i + j) = ii + ij = ii$ by the previous claim. □

Claim 15. $II = I$.

Proof. By simplicity of I, $II = 0$ or I. Since $ii = i$, the first case is impossible. □

Claim 16. $JI = 0$.

Proof. Note that JI is spanned by the set of all pairwise products of elements of J and I. Since J is a left ideal and I is a two-sided ideal, JI is a two-sided ideal. Then it can be either 0 or I by simplicity of I.

Suppose $JI = I$. Then $I = II = I(JI) = (IJ)I = 0 \cdot I = 0$, a contradiction. □

Claim 17. *J is a two-sided ideal.*

Proof. We know that it is a left ideal. For arbitrary $x \in R$, write $x = xi + xj$. Let $y \in J$ be arbitrary. Then $yx = yxi + yxj = 0 + yxj \in J$, where $yxi = 0$ since it is in JI. Thus J is also a right ideal. □

Claim 18. *$R = I \times J$ as rings.*

Proof. Let $x = x_i + x_j$ where $x_i = xi \in I$ and $xj \in J$. Similarly, let $y = y_i + y_j$. Then $xy = x_iy_i + x_iy_j + x_jy_i + x_jy_j = x_iy_i + x_jy_j$ since $x_iy_j = 0$ and $x_jy_i = 0$ by the previous claims. Thus, the map $x \mapsto (x_i, x_j)$ is a ring homomorphism.

Injective: Suppose $xi = x_j = 0$. Then $x = x1 = x(i + j) = 0$.

Surjective: let $(x_i, x_j) \in I \times J$ be arbitrary. Let $x = x_i + x_j$. Note that $x_i = x_ii + x_ij = x_ii$ by orthogonality of I and J. Similarly $x_j = x_jj$. Then $xi = (x_i + x_j)i = x_ii = x_i$ and similarly $xj = x_j$. Thus (x_i, x_j) is the image of x. □

Claim 19. *Let $K \subseteq I$ be a left I-submodule of I, where I is treated as a unital ring. Then K is a left submodule of R.*

Proof. Let $r \in R$ and $k \in K$. Then $rk = r1k = r(i + j)k = rik + rjk = rik \in K$ since $k \in K$, $ri \in I$ and $rjk = 0$ as $j \in J$ and $k \in I$ and we know $JI = 0$. Thus K is closed under left multiplication by element of R. □

Claim 20. *Both I and J are artinian (as rings).*

Proof. They are both submodules of R which is assumed to be artinian. Submodules of artinian modules are artinian. Note that R is artinian since it is semisimple. Thus they are artinian as left R-modules.

Let $K_1 \supseteq K_2 \supseteq ...$ be a descending chain of left ideals (modules) in ring I. By the previous claim, they are also left ideals in R. Since R is artinian, this chain stabilizes. But then so does $K_1 \supseteq K_2 \supseteq ....$ Thus I is artinian. Same argument applies to J. □

Claim 21. *J is (left) semisimple.*

Proof. A submodule of a semisimple module is semisimple. Thus J is semisimple as a left R-module. Let $K \subseteq J$ be a left J-submodule. Then K is a left R-submodule by the previous claim. Since R is semisimple, every submodules of a submodules has a direct complement, call it K'. Then $J = K \oplus K'$, as R-modules. Since K' is a left R-submodule, it is also a left J-submodule. Thus J is semisimple as a left J-module. □

Thus, we can repeat the process of splitting J (if it is not simple) into a direct product of simple, artinian rings. Since R is artinian, this process must stabilize, and we get a direct product of simple, artinian rings.

Apply the 29 to each of the simple, artinian rings to get the desired result. □

References

[1] Matej Brešar, *The Wedderburn-Artin Theorem*, arXiv:2405.04588 [math.RA], 2024. https://arxiv.org/abs/2405.04588.